Exam Advanced Mechanics, Wednesday, January 25 2017 from 18:30 – 22:00 in the Aletta Jacobshal 01

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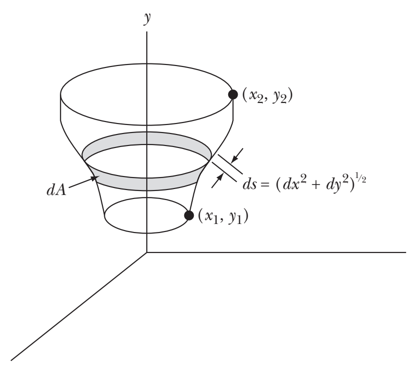
5 problems (total of 49 points).
The solution of every problem on a separate piece of paper with name and student number.
Some useful formulas are listed at the end.

Problem 1 (14 pnts in total)
Answer: Example 6.3, add fig 6-6

Consider a soap film suspended by two rings of different radii r_1 and r_2 with their centers on the y-axis, one at y_1 and the other at y_2. The surface of the rings are parallel to the $x-z$-plane. The soap-film is supposed to be massless.



2 pnts a. The soap-film is at rest. Present the physics arguments for the condition that should be imposed on the area.
Answer: minimal energy that is due to surface tension thus minimal area

2 pnts b. Express the surface area of the soap-film in terms of an integral over x and determine $f(y, y'; x)$.
Answer: $A = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}dx$

3 pnts c. Express the condition of minimal surface area in terms of an Euler equation for $y(x)$ and show that $x = a\cosh\left(\frac{y-b}{a}\right)$ is a solution.

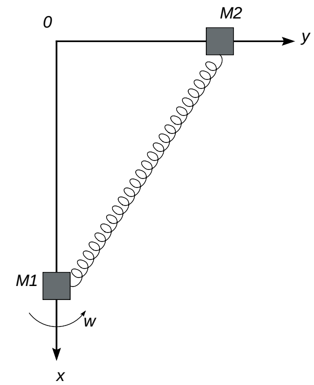
2 pnts d. Express the surface area of the soap-film in terms of an integral over y and determine $f(x, x'; y)$.
Answer: $A = 2\pi \int_{y_1}^{y_2} y\sqrt{1 + x'^2}dy$

3 pnts e. Express the condition of minimal surface area in terms of an Euler equation for $x(y)$ and show that $x = a\cosh\left(\frac{y-b}{a}\right)$ is a solution.

2 pnts f. Give the equations from which the constants a and b can be determined. DO NOT try to solve these equations!
Answer: $y_1 = Y(x_1)$ and $y_2 = Y(x_2)$

Problem 2 (13 pnts in total)
Answer: problem 7.17 or exam of Febr12, Aug10, W08; Skip 3a

A mass M_1 can move without friction on the vertical x-axis (positive = down). This mass is connected with a (massless) spring (spring constant k and unextended length l_0) to a mass M_2 that can move along the horizontal y-axis without friction. The whole system rotates with a constant angular velocity ω around the x-axis.



2 pnts a. Determine the kinetic energy and the potential energy for the system. Express the Lagrangian of the system using x and y as generalized coordinates.
Answer: $L = \frac{1}{2}M_1(\dot{x}^2 + 2g\,x) + \frac{1}{2}M_2(\dot{y}^2 + \omega^2 y^2) - \frac{1}{2}k(\sqrt{x^2+y^2} - l_0)^2$

1 pnts b. What are the constant(s) of motion?
Answer: total energy E

2 pnts c. Give the expression for the conjugated momenta, p_x and p_y, and the Hamiltonian.
Answer: $p_x = M_1\dot{x}$, $p_y = M_2\dot{y}$, $H = \frac{1}{2}p_x^2/M_1 - M_1 g\,x + \frac{1}{2}p_y^2/M_2 - M_2\omega^2 y^2/2 + \frac{1}{2}k(\sqrt{x^2+y^2} - l_0)^2$

2 pnts d. Show that the equations of motion can be written as

$$M_1\ddot{x} - M_1 g + k(d - l_0)x/d = 0 \quad ; \quad M_2\ddot{y} - M_2\omega^2 y + k(d - l_0)y/d = 0$$

where $d = \sqrt{x^2+y^2}$.

2 pnts e. Determine the two (sometimes unstable) equilibrium solutions.
Answer: $d = \sqrt{x^2+y^2}$, then $-M_1 g + k(d-l_0)x/d = 0$, &
$-M_2 y\omega^2 + k(d-l_0)y/d = 0$ giving
$y = 0$ with $d = x$ and $M_1 g = k(x - l_0)$ with $M_1 g/k + l_0 = x$
and $M_2\omega^2 d = k(d - l_0)$, $(M_2\omega^2 - k)d = -kl_0$

4 pnts f. Consider small oscillations around $y_0 = 0$ keeping x fixed at the stationary value. For what values of ω is $y_0 = 0$ a stable solution?
Answer: first order in y near $y = 0$ gives $M_2\ddot{y} - M_2\omega^2 y + k(x - l_0)y/x = 0$ substituting x_0: $M_2\omega^2 = -M_2\omega^2 + kM_1 g/(k(M_1 g/k + l_0))$ should be positive

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Problem 3 (7 pnts in total)
Answer: Example 11.10; Dumbbell

Consider a dumbbell with two equal masses m on a massless bar of length b. The dumbbell is rotating with an angular frequency ω_0 around an axis going through the center of mass and is at an angle α with the massless bar. Assume the bar to be extremely thin and the masses point-like.

1 pnts a. Make a drawing of the geometry.

1 pnts b. Calculate the inertial tensor in the body-fixed frame.
Answer: $I_1 = I_2 = 2mb^2/4 = mb^2/2$, $I_3 = 0$

2 pnts c. Calculate $\vec{L}$ in the body-fixed frame.
Answer: $\vec{\omega} = (0, \omega_0 \sin\alpha, \omega_0 \cos\alpha)$, $\vec{L} = (0, I_2\omega_0 \sin\alpha, I_3\omega_0\cos\alpha = 0)$

3 pnts d. Calculate the torque $\vec{N}$ that should be applied to the dumbbell to sustain the motion.
Answer: $\vec{N} = I\dot{\vec{\omega}} + \vec{\omega} \times \vec{L} = (I_2\omega_0^2 \sin\alpha\cos\alpha, 0, 0)$

Problem 4 (5 pnts in total)

1 pnts a. Evaluate $\partial_\mu x_\nu + \partial_\nu x_\mu$.
Answer: $= g_{\mu\nu} + g_{\mu\nu} = 2g_{\mu\nu}$

1 pnts b. Evaluate $\partial_\mu x_\nu - \partial_\nu x_\mu$.
Answer: $= g_{\mu\nu} - g_{\mu\nu} = 0$

3 pnts c. Evaluate $\partial_\mu(x^2\, x_\nu)$.
Answer: $= x_\nu \partial_\mu(x^2) + x^2 g_{\mu\nu} = 2x_\mu x_\nu + x^2 g_{\mu\nu}$

Problem 5 new (10 pnts in total)

A particle of mass m and charge e moves in a constant magnetic field whose vector potential is given by $A^0 = 0$ and $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{x})$ where $\vec{B}$ is directed along the $z-$axis.

3 pnts a. Calculate the components of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.
Answer: $A = (0, -B, B, 0)/2$ and thus $F^{12} = B$, others=zero

3 pnts b. The equation of motion of the particle for this problem is given by $\frac{d}{dt}\vec{p} = \frac{e}{c}\vec{v} \times \vec{B}$ with $\vec{p} = \gamma m \vec{v}$. Show that the energy of the particle $\epsilon = \sqrt{m^2c^2 + p^2c^2}$ remains constant during the motion.
Answer: $d\epsilon^2/dt = 2p_x\dot{p}_x + p_y\dot{p}_y = 0$

4 pnts c. At $t = 0$ the particle is at the origin, with velocity $\vec{v} = v_1\hat{x} + v_2\hat{z}$. Solve the equation of motion of the particle.
Answer: Define $\Omega = eB/(mc\gamma)$ gives $\dot{v}_x = \Omega v_y$, $\dot{v}_y = -\Omega v_x$, $\dot{v}_z = 0$, since γ is time independent because the energy is conserved. Thus $v_x = v_1 \cos\Omega t$ to obey boundary condition at t=0 and also $v_y = -v_1 \sin\Omega t$ with $v_z = v_2$

Possibly useful formulas:

$\vec{F}_B = \vec{F}_{\text{inert}} - 2m\vec{\omega} \times \vec{v}_B - m\dot{\vec{\omega}} \times \vec{r}_B - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_B)$, and $\vec{v}_I = \vec{v}_B + \vec{\omega} \times \vec{r}_B$

The response of a damped oscillator $\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t)/m$ to a delta force at $t = 0$ is $\frac{1}{\omega_1 m} e^{-\beta t} \sin\omega_1 t$ for $t > 0$, where $\omega_1 = \sqrt{\omega_r^2 - \beta^2}$.

The 'alternative' form of the Euler equation for $f(y, y'; x)$ is

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right)$$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\quad \cos(\alpha - \beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$

$\vec{B} = \vec{\nabla} \times \vec{A}$; $\quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial ct}$

Integrals

For $c > 0$ we have:

$$\int e^{cx} dx = \frac{1}{c} e^{cx} \; ; \quad \int x\, e^{cx} dx = \frac{cx - 1}{c^2} e^{cx} \; ; \quad \int x^2 e^{cx} dx = \frac{c^2x^2 - 2cx + 2}{c^3} e^{cx}$$